Enhancing Credit Default Swaps Pricing with Meshfree Methods

CCFEA
Workshop

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Introduction

- CDSs: the most popular credit derivatives in the market.

- They contain information of the default probability.
  - Accurate approximation of the credit risk implicit in the CDS spreads is essential in pricing similar derivatives.

- Models:
  - Multi-factor stochastic processes with closed-form solution
  - Stochastic processes with no closed-form solution. Numerical methods have to be implemented.

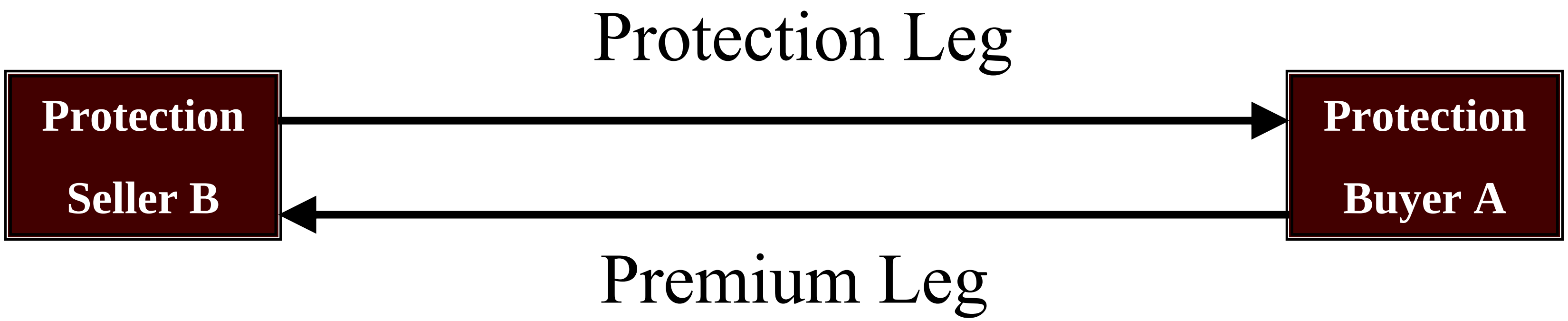
- Numerical approach has become a basic tool in financial engineering
  - If there is no closed-form solution or
  - If a closed-form solution is based on strong assumptions.

Credit Default Swap (CDS) Payoff

- A CDS is a contract which ensures protection against default of a reference credit. Companies "A" and "B" agree that:

    1. If a company "C" (Reference Credit) defaults at time $\tau = \tau_C$, with $T_a < \tau \leq T_b$, then "B" pays to "A" a certain cash amount L_{GD}.

    2. In exchange, "A" pays to "B" a rate R at times $T_{a+1},\ldots,T_b$ or until default τ_C.



CDS Pricing

- **CDS Payoff = Premium Leg – Protection Leg**

- In equilibrium **CDS Payoff = 0**, and therefore

$$R_{a,b} = \frac{-L_{GD}\left[\int_{t=T_a}^{T_b} P(0,t)\, d_t\mathbb{Q}(\tau \geq t)\right]}{-\int_{t=T_a}^{T_b} P(0,t)\left(t - T_{\beta(t)-1}\right) d_t\mathbb{Q}(\tau \geq t) + \sum_{i=a+1}^{b} P(0,T_i)\,\alpha_i\mathbb{Q}(\tau \geq T_i)}$$

- CDS Spread is a function of …

Discount Factor

$$P(0,t)$$

Survival Probability

$$\mathbb{Q}(\tau \geq t)$$

Stochastic Setting

- We assume:
  1. A stochastic setting for r_t

$$P\left(0,t\right)=\mathbb{E}\left[D\left(0,t\right)\right]=\mathbb{E}\left(\exp^{-\int_0^t r_s ds}\right)$$

  2. The default time τ is the first jump of a Cox process with stochastic intensity λ_t

$$\mathbb{Q}\left(\tau\geq t\right)=\mathbb{E}\left[\exp^{-\int_0^t \lambda_s ds}\right]$$

Stochastic Setting

- We consider:
  1. r_t follows a multi-factor Cox-Ingersoll-Ross process.

$$r_t = \sum_{i=1}^{N} x_{i,t} \qquad dx_{i,t} = \kappa_i \left(\theta_i - x_{i,t}\right) dt + \sigma_i \sqrt{x_{i,t}} dW_{i,t},$$

$$\rho_{i,j} = \mathbb{E}\left[dW_{i,t}, dW_{j,t}\right]$$

  2. λ_t follows a multi-factor Black-Karasinski process.

$$\lambda_t = \sum_{i=1}^{M} \exp^{y_{i,t}} \qquad dy_{i,t} = \eta_i \left(\ln \mu_i - y_{i,t}\right) dt + \nu_i dW_{i,t},$$

$$\gamma_{i,j} = \mathbb{E}\left[dW_{i,t}, dW_{j,t}\right]$$

Problems and Solutions

- **Problems:**

  1. There is no closed-form solution for the BK model.

  2. The closed-form solution for the Multi-Factor CIR model requires that $E\left[dW_{i,t}, dW_{j,t}\right] = \rho_{i,j} = 0$

- **Solution:**

  1. Approximation by Numerical Methods.

Numerical Approximation

- The solution for $P(0,t)$ by Feynman-Kac theorem is given by the PDE

$$\frac{\partial P}{\partial t}+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{i,j}\sigma_i\sigma_j\sqrt{x_i}\sqrt{x_j}\frac{\partial^2 P}{\partial x_i\partial x_j}+\sum_{i=1}^{N}\kappa_i\left(\theta_i-x_i\right)\frac{\partial P}{\partial x_i}-\left[\sum_{i=1}^{N}x_i\right]P=0$$

- The solution for $\mathbb{Q}\left(\tau\geq t\right)$ by Feynman-Kac theorem is given by the PDE

$$\frac{\partial \mathbb{Q}}{\partial t}+\frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M}\gamma_{i,j}v_iv_j\frac{\partial^2 \mathbb{Q}}{\partial y_i\partial y_j}+\sum_{i=1}^{M}\eta_i\left(\ln\mu_i-y_i\right)\frac{\partial \mathbb{Q}}{\partial y_i}-\left[\sum_{i=1}^{M}\exp^{y_i}\right]\mathbb{Q}=0$$

Finite Difference Methods

- One of the most popular techniques for approximating PDE in finance are: **Finite Difference Methods (FDM)**
    - FDM work well in simple problems (1 dimension)
    - However, in more complex applications with higher dimensions:
        - FDM face difficulties associated with
            1. The discretization
            2. The underlying grid and
            3. Regularity conditions.
        - The computational complexity in the construction of a fixed grid grows exponentially with the dimension

Meshfree Methods

- Meshfree Methods (MFM) are a collection of modern techniques for numerical approximation.

  - MFM do not use grids.

  - They use a set of independent points scattered over the domain of the problem

  - These methods avoid the implicit cost in the mesh generation

  - MFM deal with
    - Complex geometries
    - Irregular discretization
    - High dimensional problems with great accuracy

  - **Key point:** MFM work well in multi-dimensional problems

Radial Basis Function Interpolation

- RBF interpolation uses a set of quasi-random points over the space
- This approach deals with univariate basis functions and the Euclidean norm
- RBF interpolation changes a multi-dimensional problem into a one-dimensional problem
- Its numerical results offer a highly accurate spatial approximation
- It easily deals with the correlation terms in multi-factor problems

Radial Basis Function Interpolation

- Radial Basis Function (RBF) Interpolation approximates the value of the function as the weighted sum of RBFs evaluated on a set of points

$$f(\mathbf{z},t) \simeq \sum_{k=1}^{K} \delta_k(t)\,\varphi(\|\ \mathbf{z},\mathbf{z}_k\ \|)$$

where : $\delta_k(t)$ are the unknown weights

$\varphi(\cdot)$ is the chosen radial basis function

$\|\ \mathbf{z},\mathbf{z}_k\ \|$ is the Euclidean norm

$\mathbf{z} = [\mathbf{z}_1,\ldots,\mathbf{z}_K]'$ is a vector of points

- One of the most popular RBF is

$$\text{Thin Plate Spline (TPS)} \quad : \quad \varphi(\mathbf{z},\mathbf{z}_k) =: \varphi(r_k) = r_k^4 \log(r_k)$$

where $r_k = \|\ \mathbf{z}-\mathbf{z}_k\ \|$ is the Euclidean norm.

Experiments

- We apply radial basis function (RBF) interpolation to numerically approximate:
  - Zero-coupon bond price: one-factor and two-factor CIR model
  - Analysis of volatility effects on numerical approximations of zero-coupon bond prices
  - Survival Probability : one-factor and two-factor BK model
  - Efficiency analysis of numerical approaches of survival probability
  - CDS Spreads
    1. One-factor model
    2. Two-factor models
    3. Two-factor models under correlation structures

Experiments

- Zero-coupon bond price
  - Parameters $dx_{i,t} = \kappa_i (\theta_i - x_{i,t})\, dt + \sigma_i \sqrt{x_{i,t}} dW_{i,t},$

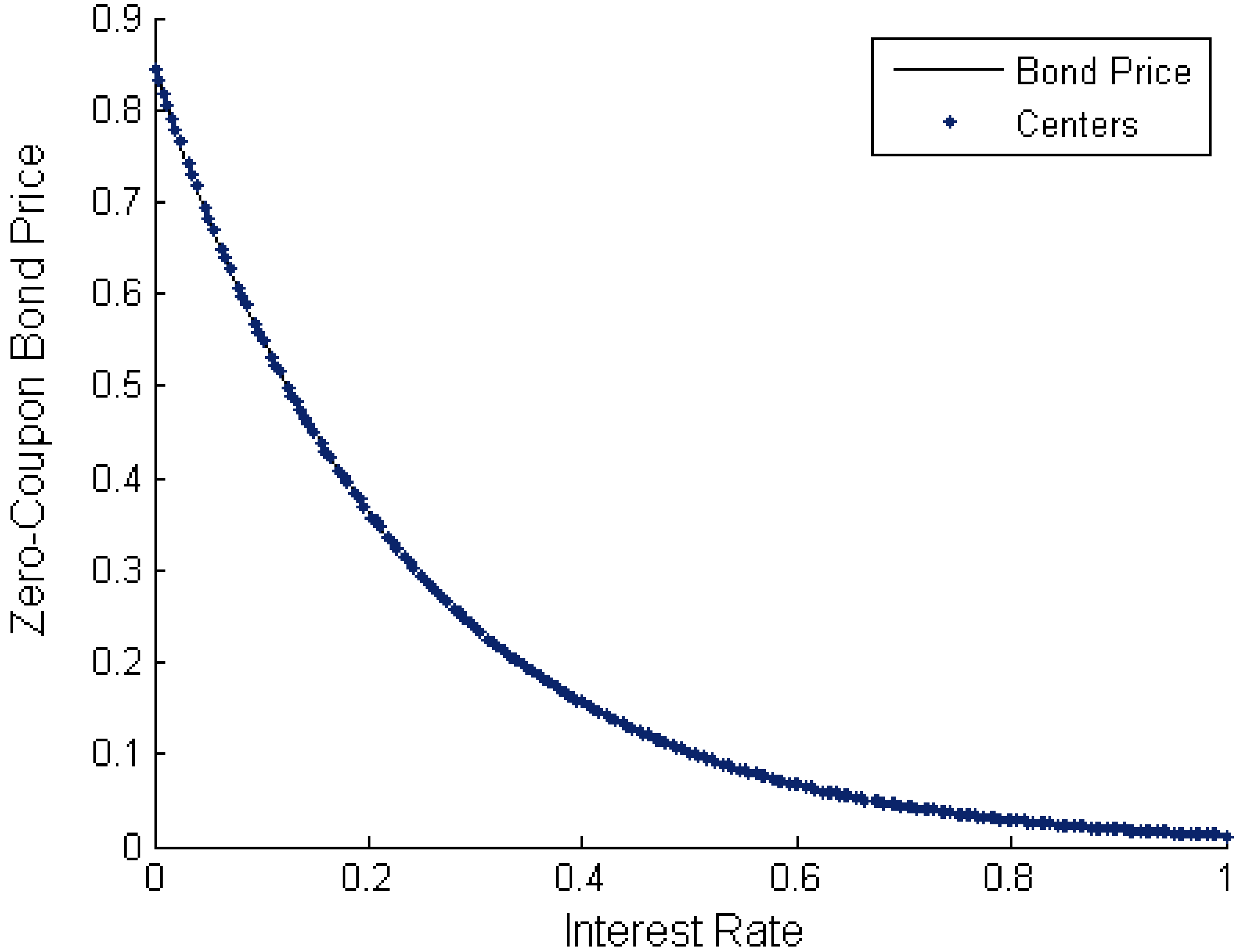
    $\kappa_1 = 0.2,\ \theta_1 = 0.03,\ \sigma_1 = 0.06,\ \kappa_2 = 0.1,\ \theta_2 = 0.02,\ \sigma_2 = 0.05$

    $\rho_{1.2} = \rho_{2.1} = 0$

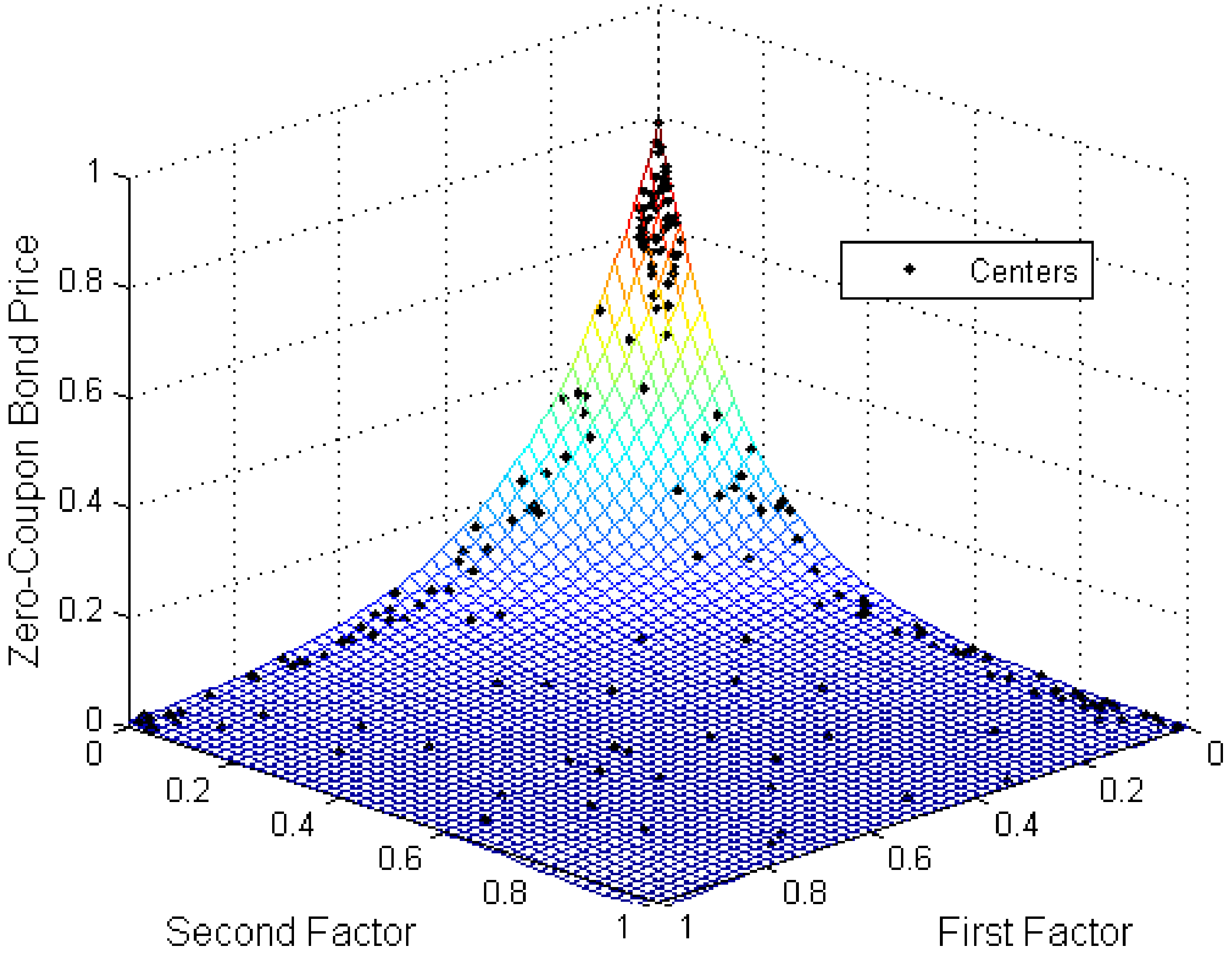
  - RBF Interpolation : 200 centers and 200 time steps
    - One Factor: Halton Collocation in a spatial domain $[0, 1]$
    - Two Factors: 4 groups of 50 points each under a uniform distribution over the spatial domains $[0, 0.05]^2$, $[0, 0.05] \times [0, 1]$, $[0, 1] \times [0, 0.05]$ and $[0, 1]^2$

  - FDM:
    - One Factor: $N_x \times N_t$, with $N_x = 1000$ and $N_t = 200$
    - Two Factors: $N_{x_1} \times N_{x_2} \times N_t$ with $N_{x_1} = 500$, $N_{x_2} = 500$ and $N_t = 200$

Zero-Coupon Bond Price: One-Factor CIR Model



Zero-Coupon Bond Price: Two-Factor CIR Model



Zero-Coupon Bond Price: One-Factor CIR Model

Panel A: Analytical Solution

$x_{1,0}$	1Y	3Y	5Y	7Y	10Y
0.020	0.9793	0.9350	0.8891	0.8433	0.7767
0.025	0.9749	0.9245	0.8753	0.8278	0.7604
0.030	0.9705	0.9142	0.8617	0.8125	0.7445
0.035	0.9661	0.9040	0.8483	0.7976	0.7290
0.040	0.9617	0.8939	0.8351	0.7829	0.7137

Panel B: Radial Basis Function

$x_{1,0}$	1Y	3Y	5Y	7Y	10Y
0.020	0.9793	0.9350	0.8891	0.8433	0.7767
0.025	0.9749	0.9245	0.8753	0.8278	0.7604
0.030	0.9705	0.9142	0.8617	0.8125	0.7445
0.035	0.9661	0.9040	0.8483	0.7976	0.7290
0.040	0.9617	0.8939	0.8351	0.7829	0.7137
RMSE	**1.2E-06**	**2.8E-06**	**3.9E-06**	**4.9E-06**	**6.0E-06**

Zero-Coupon Bond Price: One-Factor CIR Model

Panel A: Analytical Solution

$x_{1,0}$	1Y	3Y	5Y	7Y	10Y
0.020	0.9793	0.9350	0.8891	0.8433	0.7767
0.025	0.9749	0.9245	0.8753	0.8278	0.7604
0.030	0.9705	0.9142	0.8617	0.8125	0.7445
0.035	0.9661	0.9040	0.8483	0.7976	0.7290
0.040	0.9617	0.8939	0.8351	0.7829	0.7137

Panel C: Finite Difference Method

$x_{1,0}$	1Y	3Y	5Y	7Y	10Y
0.020	0.9793	0.9350	0.8891	0.8433	0.7768
0.025	0.9749	0.9245	0.8753	0.8278	0.7605
0.030	0.9705	0.9142	0.8617	0.8126	0.7446
0.035	0.9661	0.9040	0.8483	0.7976	0.7290
0.040	0.9617	0.8939	0.8351	0.7829	0.7138
RMSE	**6.8E-07**	**4.5E-06**	**6.7E-06**	**3.5E-05**	**1.6E-04**

Zero-Coupon Bond Price: Two-Factor CIR Model

Panel A: Analytical Solution

$x_{1,0}$	$x_{2,0}$	1Y	3Y	5Y	7Y	10Y
0.020	0.010	0.9691	0.9038	0.8371	0.7717	0.6791
0.025	0.015	0.9601	0.8822	0.8082	0.7390	0.6447
0.030	0.020	0.9513	0.8611	0.7802	0.7076	0.6121
0.035	0.025	0.9425	0.8406	0.7533	0.6776	0.5811
0.040	0.030	0.9337	0.8205	0.7272	0.6488	0.5516

Panel B: Radial Basis Function

$x_{1,0}$	$x_{2,0}$	1Y	3Y	5Y	7Y	10Y
0.020	0.010	0.9691	0.9038	0.8371	0.7717	0.6791
0.025	0.015	0.9601	0.8822	0.8082	0.7390	0.6447
0.030	0.020	0.9513	0.8611	0.7802	0.7076	0.6121
0.035	0.025	0.9425	0.8406	0.7533	0.6776	0.5811
0.040	0.030	0.9337	0.8205	0.7272	0.6488	0.5516
	RMSE	**3.3E-06**	**3.1E-06**	**1.7E-06**	**1.1E-06**	**4.0E-06**

Zero-Coupon Bond Price: Two-Factor CIR Model

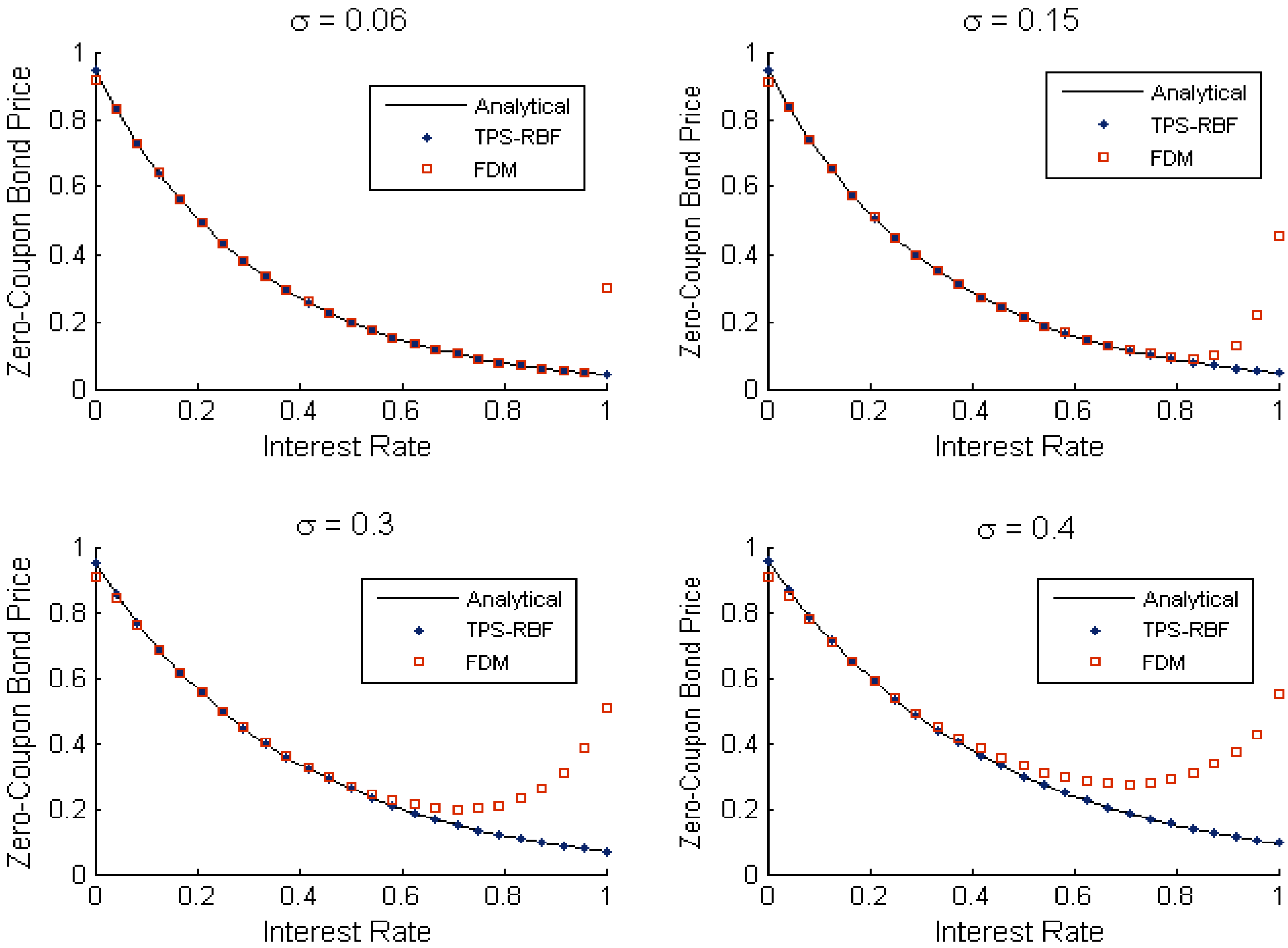
Panel A: Analytical Solution

$x_{1,0}$	$x_{2,0}$	1Y	3Y	5Y	7Y	10Y
0.020	0.010	0.9691	0.9038	0.8371	0.7717	0.6791
0.025	0.015	0.9601	0.8822	0.8082	0.7390	0.6447
0.030	0.020	0.9513	0.8611	0.7802	0.7076	0.6121
0.035	0.025	0.9425	0.8406	0.7533	0.6776	0.5811
0.040	0.030	0.9337	0.8205	0.7272	0.6488	0.5516

Panel C: Finite Difference Method

$x_{1,0}$	$x_{2,0}$	1Y	3Y	5Y	7Y	10Y
0.020	0.010	0.9688	0.9019	0.8331	0.7661	0.6730
0.025	0.015	0.9599	0.8804	0.8042	0.7330	0.6369
0.030	0.020	0.9510	0.8595	0.7765	0.7019	0.6043
0.035	0.025	0.9423	0.8391	0.7499	0.6724	0.5740
0.040	0.030	0.9335	0.8191	0.7242	0.6442	0.5454
	RMSE	**2.4E-04**	**1.6E-03**	**3.4E-03**	**4.9E-03**	**7.1E-03**

Zero-Coupon Bond Price with Different Values for the Volatility



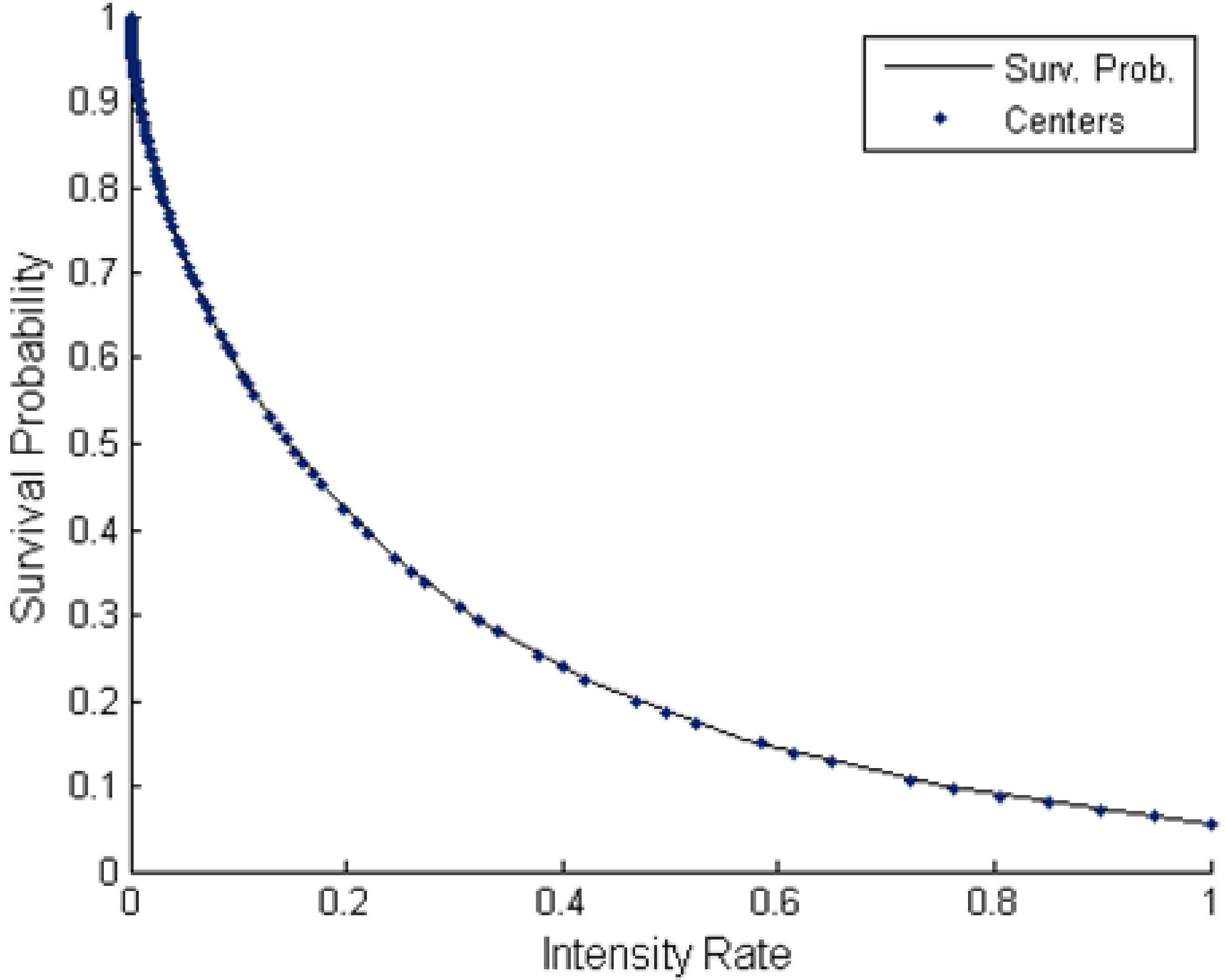
Experiments

- Survival Probability
  - Parameters $dy_{i,t} = \eta_i \left(\ln \mu_i - y_{i,t}\right) dt + \nu_i dW_{i,t}.$

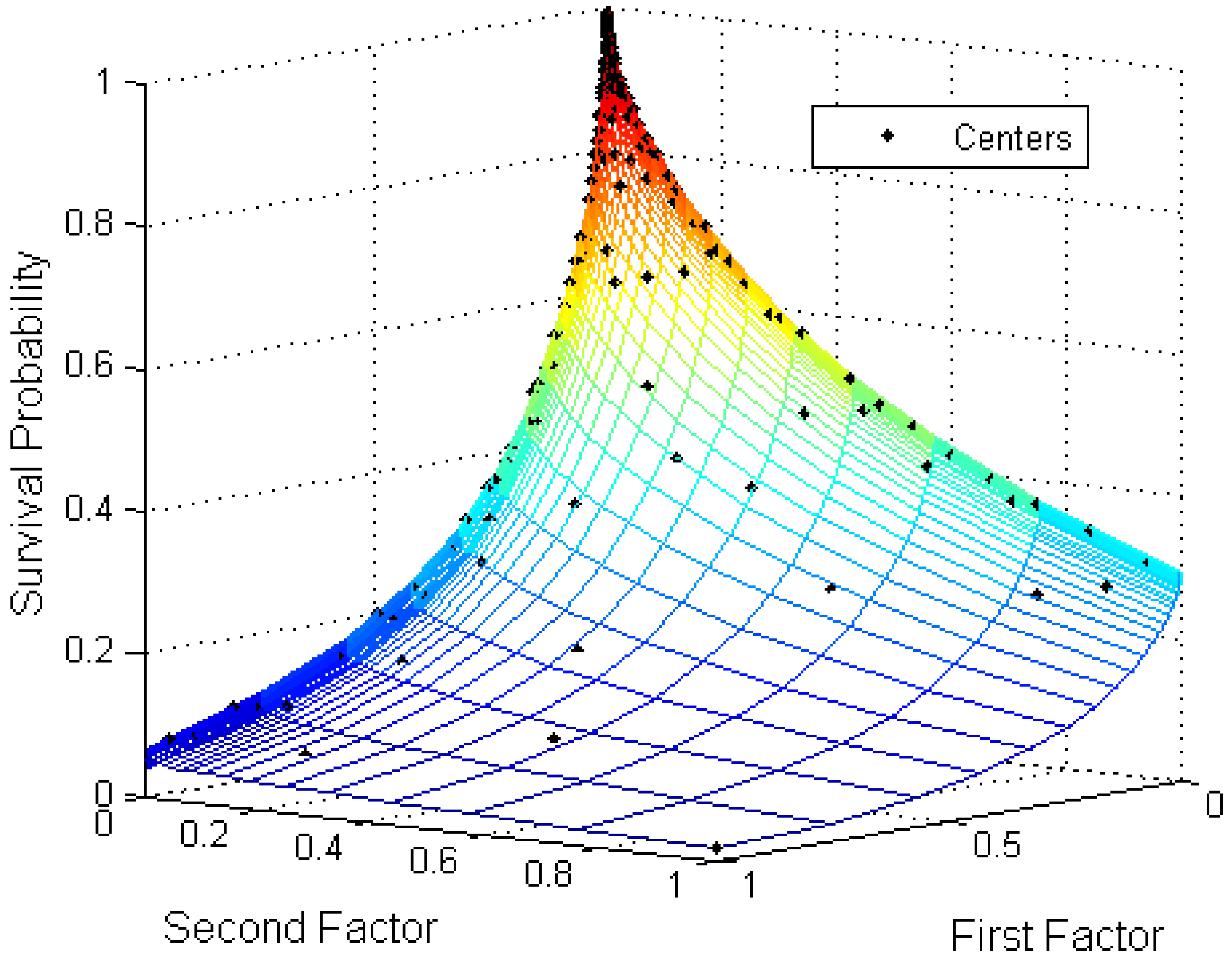
    $\eta_1 = 0.1,\ \mu_1 = 0.015,\ \nu_1 = 0.06,\ \eta_2 = 0.2,\ \mu_2 = 0.005$ and $\nu_2 = 0.08$

    $\gamma_{1,2} = \gamma_{2,1} = 0.$
  - RBF Interpolation : 200 centers and 200 time steps
    - One Factor: Halton Collocation in a spatial domain $[0, 1]$
    - Two Factors: Halton Collocation in a spatial domain $[0, 1]^2$
  - FDM:
    - One Factor: $N_x \times N_t$, with $N_x = 1000$ and $N_t = 200$
    - Two Factors: $N_{x_1} \times N_{x_2} \times N_t$ with $N_{x_1} = 500$, $N_{x_2} = 500$ and $N_t = 200$

Survival Probability: One-Factor BK Model



Survival Probability: Two-Factor BK Model



Survival Probability: One-Factor BK Model

Panel A: Radial Basis Function

$y_{1,0}$	1Y	3Y	5Y	7Y	10Y
0.003	0.9968	0.9888	0.9787	0.9665	0.9448
0.005	0.9947	0.9826	0.9686	0.9527	0.9261
0.010	0.9898	0.9687	0.9467	0.9241	0.8896
0.015	0.9851	0.9559	0.9275	0.9000	0.8601
0.020	0.9805	0.9438	0.9099	0.8783	0.8345
0.025	0.9759	0.9322	0.8935	0.8585	0.8114

Panel B: Finite Difference Method

$y_{1,0}$	1Y	3Y	5Y	7Y	10Y
0.003	0.9968	0.9888	0.9787	0.9665	0.9448
0.005	0.9947	0.9826	0.9686	0.9527	0.9261
0.010	0.9898	0.9687	0.9467	0.9241	0.8896
0.015	0.9851	0.9559	0.9275	0.9000	0.8601
0.020	0.9805	0.9438	0.9099	0.8783	0.8345
0.025	0.9759	0.9322	0.8935	0.8585	0.8114
RMSE	**3.2E-07**	**2.4E-06**	**5.8E-06**	**1.0E-05**	**1.8E-05**

Survival Probability: Two-Factor BK Model

Panel A: Radial Basis Function

$y_{1,0}$	$y_{2,0}$	1Y	3Y	5Y	7Y	10Y
0.003	0.001	0.9958	0.9848	0.9704	0.9530	0.9217
0.005	0.001	0.9936	0.9781	0.9593	0.9378	0.9013
0.010	0.003	0.9867	0.9588	0.9294	0.8994	0.8540
0.015	0.005	0.9802	0.9415	0.9040	0.8681	0.8170
0.020	0.008	0.9731	0.9237	0.8790	0.8382	0.7829
0.025	0.010	0.9669	0.9088	0.8583	0.8136	0.7550

Panel B: Finite Difference Method

$y_{1,0}$	$y_{2,0}$	1Y	3Y	5Y	7Y	10Y
0.003	0.0008	0.9959	0.9855	0.9725	0.9570	0.9298
0.005	0.0010	0.9937	0.9795	0.9632	0.9451	0.9151
0.010	0.0030	0.9871	0.9617	0.9367	0.9120	0.8754
0.015	0.0050	0.9808	0.9461	0.9148	0.8859	0.8458
0.020	0.0080	0.9737	0.9294	0.8922	0.8597	0.8166
0.025	0.0100	0.9678	0.9159	0.8746	0.8398	0.7950
	RMSE	**0.0005**	**0.0044**	**0.0103**	**0.0168**	**0.0267**

Survival Probability: Efficiency Comparison between Methods

Panel A: One-Factor BK Model

RBF		**FDM**	
s = 200		**Nt = 200**	
Centers	**RMSE**	**Nx**	**RMSE**
500		500	1.2E-05
400	1.0E-09	400	1.5E-05
300	1.3E-09	300	2.0E-05
200	1.8E-09	200	3.2E-05
100	2.5E-08	100	7.1E-05
80	4.2E-07	80	1.0E-04
50	1.1E-05	50	2.3E-04

Panel B: Two-Factor BK Model

RBF		**FDM**	
s = 200		**Nt = 200**	
Centers	**RMSE**	**Nx**	**RMSE**
500		500 x 500	9.79E-03
400	1.2E-04	400 x 400	9.80E-03
300	2.1E-04	300 x 300	9.80E-03
200	5.0E-04	200 x 200	9.81E-03
100	7.5E-04	100 x 100	9.83E-03
80	1.5E-03	80 x 80	9.83E-03
50	5.1E-03	50 x 50	9.84E-03

CDS Spreads: One-Factor CIR Model, One-Factor BK Model

Panel A: Radial Basis Function

$x_{1,0}$	$y_{1,0}$	1Y	3Y	5Y	7Y	10Y
0.015	0.003	19.5	22.6	25.8	28.9	33.4
0.020	0.005	31.7	35.0	38.2	41.2	45.3
0.025	0.010	61.4	63.7	65.7	67.5	69.8
0.030	0.015	90.4	90.5	90.6	90.6	90.7
0.035	0.020	118.9	116.2	114.0	112.1	109.8

Panel B: Finite Difference Method

$x_{1,0}$	$y_{1,0}$	1Y	3Y	5Y	7Y	10Y
0.015	0.003	19.5	22.6	25.8	29.0	33.5
0.020	0.005	31.8	35.1	38.3	41.3	45.4
0.025	0.010	61.4	63.7	65.8	67.6	69.9
0.030	0.015	90.4	90.5	90.6	90.7	90.7
0.035	0.020	119.0	116.2	114.0	112.1	109.7
	RMSE	**0.03**	**0.03**	**0.04**	**0.05**	**0.06**

CDS Spreads: Two-Factor CIR Model, Two-Factor BK Model

Panel A: Radial Basis Function

$x_{1,0}$	$x_{2,0}$	$y_{1,0}$	$y_{2,0}$	1Y	3Y	5Y	7Y	10Y
0.015	0.005	0.003	0.0008	25.3	30.6	35.8	40.5	47.3
0.020	0.010	0.005	0.0010	38.8	44.3	49.5	54.2	60.5
0.025	0.015	0.010	0.0030	80.5	84.4	87.8	90.5	93.8
0.030	0.020	0.015	0.0050	120.5	121.2	121.7	121.9	121.9
0.035	0.025	0.020	0.0080	165.0	160.1	156.5	153.5	149.9

Panel B: Finite Difference Method

$x_{1,0}$	$x_{2,0}$	$y_{1,0}$	$y_{2,0}$	1Y	3Y	5Y	7Y	10Y
0.015	0.005	0.003	0.0008	24.9	29.1	33.3	37.2	42.5
0.020	0.010	0.005	0.0010	38.0	41.5	44.9	48.0	52.2
0.025	0.015	0.010	0.0030	78.4	78.4	78.8	79.2	80.0
0.030	0.020	0.015	0.0050	117.2	111.8	108.0	105.3	102.5
0.035	0.025	0.020	0.0080	160.9	148.2	139.4	133.0	126.4
			RMSE	**3.2**	**9.2**	**13.2**	**15.9**	**18.4**

CDS Spreads: Two-Factor CIR Model, Two-Factor BK Model

Panel C: Differences (FDM-RBF)

$x_{1,0}$	$x_{2,0}$	$y_{1,0}$	$y_{2,0}$	1Y	3Y	5Y	7Y	10Y
0.015	0.005	0.003	0.0008	-0.4	-1.5	-2.5	-3.3	-4.8
0.020	0.010	0.005	0.0010	-0.9	-2.8	-4.6	-6.2	-8.4
0.025	0.015	0.010	0.0030	-2.1	-6.0	-9.0	-11.3	-13.8
0.030	0.020	0.015	0.0050	-3.2	-9.4	-13.7	-16.5	-19.3
0.035	0.025	0.020	0.0080	-4.1	-12.0	-17.1	-20.4	-23.5

CDS Spreads under Correlation Structures

Panel A: Positive Correlation

$x_{1,0}$	$x_{2,0}$	$y_{1,0}$	$y_{2,0}$	1Y	3Y	5Y	7Y	10Y
0.015	0.005	0.003	0.0008	25.3	30.6	35.8	40.5	47.3
0.020	0.010	0.005	0.0010	38.8	44.3	49.5	54.2	60.6
0.025	0.015	0.010	0.0030	80.5	84.4	87.8	90.5	93.8
0.030	0.020	0.015	0.0050	120.5	121.2	121.7	121.8	121.8
0.035	0.025	0.020	0.0080	165.0	160.1	156.5	153.4	149.8
			RMSE	**0.002**	**0.008**	**0.022**	**0.032**	**0.044**

Panel B: Negative Correlation

$x_{1,0}$	$x_{2,0}$	$y_{1,0}$	$y_{2,0}$	1Y	3Y	5Y	7Y	10Y
0.015	0.005	0.003	0.0008	25.3	30.6	35.8	40.5	47.3
0.020	0.010	0.005	0.0010	38.8	44.3	49.5	54.2	60.5
0.025	0.015	0.010	0.0030	80.5	84.4	87.8	90.5	93.8
0.030	0.020	0.015	0.0050	120.5	121.2	121.7	121.9	121.9
0.035	0.025	0.020	0.0080	165.0	160.1	156.5	153.5	150.0
			RMSE	**0.004**	**0.017**	**0.045**	**0.065**	**0.088**

CDS Spreads under Correlation Structures

Panel C: Mixed Correlation

$x_{1,0}$	$x_{2,0}$	$y_{1,0}$	$y_{2,0}$	1Y	3Y	5Y	7Y	10Y
0.015	0.005	0.003	0.0008	25.3	30.6	35.8	40.5	47.3
0.020	0.010	0.005	0.0010	38.8	44.3	49.5	54.2	60.5
0.025	0.015	0.010	0.0030	80.5	84.4	87.8	90.5	93.8
0.030	0.020	0.015	0.0050	120.5	121.2	121.7	121.9	121.9
0.035	0.025	0.020	0.0080	165.0	160.1	156.5	153.5	149.9
			RMSE	**0.004**	**0.014**	**0.037**	**0.050**	**0.056**

Conclusions

- The results show that:

  - RBF approach is computationally efficient and achieve more accurate and stable results than FDM.

  - CDS spreads present considerable differences between both approaches for two-factor models, which are bigger for instruments with a long maturity.

  - Correlation between factor does not have decisive effects on the CDS spreads.