Moment optimization for asset allocation with pair-copula decompositions

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Contents

Motivation I

- The traditional Markowitz theory assumes that asset returns are multivariate normally distributed, which contraries to empirical evidence.
- Knowledge of the dependency structure between underlying assets has become increasingly important in major financial applications such as portfolio management. Diversification effect may be overstated if portfolio managers do not consider the nonlinear and asymmetrical dependency structure.
- Most of the copula applications are focused on modelling bivariate distributions, copulae also have been drawn attention to modelling high-dimensional distributions. However, the literature about high-dimensional copula applications is limited.

Motivation II

- There has been a proliferation of studies about bivariate copula applications in portfolio construction, such as utility maximization [Patton, 2004], VaR or CVaR [Embrechts et al., 2002] and [Bradley and Taqqu, 2004].
- In addition to optimizing the utility functions and the return quantile functions, performance indicators such as the Omega ratio, the maximum loss and the Sharpe ratio (see [Gilli et al., 2006] and [Kopman and Liu, 2009]) have been employed as optimization criteria to distribute asset weights. However, none of these asset allocation studies was considered asymmetric dependency of asset returns.

Motivation III

- This paper addresses the asset allocation problem by optimizing a portfolio performance measure, with the goal of improving portfolio performance by taking the asymmetric dependency of high-dimensional asset returns into consideration.
- A pair-copula TGARCH model is proposed to model the high-dimensional dependency structure of asset returns. As a recent study from [Fischer et al., 2009] shows that the vine pair-copula construction which takes bivariate copulae as building blocks may be more appropriate in modelling high-dimensional distributions than the approaches including the multivariate Archimedean copulae, the Koehler-Symanowski copulae and the Multiplicative Liebscher copulae.

Motivation IV

- The proposed asset allocation model optimizes portfolio weights by maximizing a portfolio performance measure. Three measures, the Omega ratio, the Sharpe ratio and the modified Sharpe ratio are considered as candidates in comparing the portfolio performance and finding a suitable criterion.

The pair-copula I

- Considering a vector $\mathbf{X} = (X_1, \ldots, X_d)$ has the follow joint density distribution function ([Bedford and Cooke, 2002])

$$f(x_1, \ldots, x_d) = f(x_d) \cdot f(x_{d-1}|x_d) \cdot f(x_{d-2}|x_{d-1}, x_d) \cdot \ldots f(x_1|x_2, \ldots, x_d). \tag{1}$$

- According to [Sklar, 1959], any multivariate distributions F with marginal densities $F_1(x_1), \ldots, F_d(x_d)$ may be written as

$$F(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\}. \tag{2}$$

- Therefore, the copula in Eq. (2) can be written as

$$C(u_1, \ldots, u_d) = F\{F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\}. \tag{3}$$

The pair-copula II

- By applying the chain rule, one has the joint density of Eq. (3)

$$f(x_1,\ldots,x_d) = c_{1,\ldots,d}\{F_1(x_1),\ldots,F_d(x_d)\}\cdot f_1(x_1)\cdots f_d(x_d). \tag{4}$$

- One can further obtain the conditional densities. In the bivariate case, the conditional density can be found as

$$f(x_1|x_2) = c_{1,2}\{F_1(x_1),F_2(x_2)\}\cdot f_1(x_1). \tag{5}$$

- Apparently, the term in Eq. (1) can be decomposed into two parts, the pair-copula and the conditional marginal density. The latter has a general expression:

$$f(x|\mathbf{v}) = c_{x,v_j|\mathbf{v}_{-j}}\{F(x|\mathbf{v}_{-j}),F(v_j|\mathbf{v}_{-j})\}\cdot f(x|\mathbf{v}_{-j}). \tag{6}$$

The pair-copula III

- [Joe, 1996] showed that the $F(x|\mathbf{v})$ term can be computed by:

$$F(x|\mathbf{v}) = \frac{\partial C_{x,v_j|\mathbf{v}_{-j}}\{F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j})\}}{\partial F(v_j|\mathbf{v}_{-j})}. \tag{7}$$

- However, the pair-copula decompositions in high-dimensional cases (e.g. $d \geq 3$) are not unique. For example, one can have two possible pair-copula densities for a 3-dimensional case:

$$f(x_1|x_2, x_3) = c_{1,2|3}\{F(x_1|x_3), F(x_2|x_3)\} \cdot f(x_1|x_3), \quad \text{or} \tag{8}$$

$$f(x_1|x_2, x_3) = c_{1,3|2}\{F(x_1|x_2), F(x_3|x_2)\} \cdot f(x_1|x_2). \tag{9}$$

The pair-copula IV

In order to have an unique decomposition, the so-called vines, e.g. the D-vine and the canonical vine are introduced by [Kurowicka and Cooke, 2007] to describe the decomposition schemes graphically.
Since the D-vine structure has been recently discussed and recommended by researchers for high-dimensional distribution modelling (see [Aas et al., 2007] and [Fischer et al., 2009]), we also adopt the D-vine structure in this study. The D-vine has a tree structure as shown in the next figure.

The pair-copula V

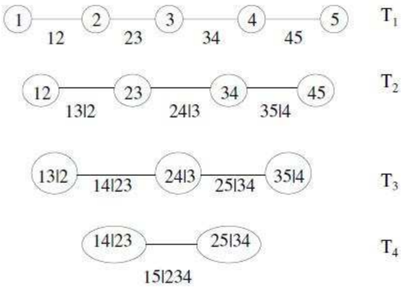


Figure: Five-Dimensional D-vine Structure (adopted from [Aas et al., 2007])

The pair-copula VI

- In the 5-dimensional case, the copula density with the D-vine decomposition can be written as

$$f(x_1,x_2,x_3,x_4,x_5) = \\ f_5(x_5)\cdot f(x_4|x_5)\cdot f(x_3|x_4,x_5)\cdot f(x_2|x_3,x_4,x_5)\cdot f(x_1|x_2,x_3,x_4,x_5). \tag{10}$$

The pair-copula VII

- By applying the decomposition, we have

$$
\begin{aligned}
f(x_4|x_5) &= c_{45}\{F_4(x_4), F_5(x_5)\} \cdot f_4(x_4) && (11)\\
f(x_3|x_4,x_5) &= c_{35|4}\{F(x_3|x_4), F(x_5|x_4)\} \\
&\quad \cdot c_{34}\{F_3(x_3), F_4(x_4)\} \cdot f_3(x_3) && (12)\\
f(x_2|x_3,x_4,x_5) &= c_{25|34}\{F(x_2|x_3,x_4), F(x_5|x_3,x_4)\} \\
&\quad \cdot c_{24|3}\{F(x_2|x_3), F(x_4|x_3)\} \cdot c_{23}\{F_2(x_2), F_3(x_3)\} \cdot f_2(x_2) && (13)\\
f(x_1|x_2,x_3,x_4,x_5) &= c_{15|234}\{F(x_1|x_2,x_3,x_4), F(x_5|x_2,x_3,x_4)\} \\
&\quad \cdot c_{14|23}\{F(x_1|x_2,x_3), F(x_4|x_2,x_3)\} \\
&\quad \cdot c_{13|2}\{F(x_1|x_2), F(x_3|x_2)\} \cdot c_{12}\{F_1(x_1), F_2(x_2)\} \cdot f_1(x_1). && (14)
\end{aligned}
$$

The conditional cumulative distributions can be further decomposed by applying Eq. (7).

The pair-copula VIII

- An important issue should be considered while using the vine copulae, i.e. the choice of pair-copula types.
  [Fischer et al., 2009] concluded that the bivariate Student t copula should be considered as the first choice, since the copula behaves better than other elliptical copulae and the Archimedean copulae according to goodness-of-fit (GoF) tests.

$$c^t_{\nu,\rho}(u_1,u_2) = \frac{1}{\sqrt{|\rho|}} \frac{\Gamma(\frac{\nu+2}{2})\Gamma(\frac{\nu}{2})^{n-1}}{\Gamma(\frac{\nu+1}{2})^2} \frac{\prod_{k=1}^2 (1+\frac{y_k^2}{\nu})^{\frac{\nu+1}{2}}}{(1+\frac{y'\rho^{-1}y}{\nu})^{\frac{\nu+2}{2}}}. \quad (15)$$

The pair-copula IX

- Only the one-parameter Archimedean copulae were included in the study of [Fischer et al., 2009], considering the mixtures of Archimedean copulae or the two-parameter Archimedean families may further justify the study. Therefore, we consider the Joe-Clayton copula:

$$c^{jc}(u_1, u_2; \tau^U, \tau^L) = \frac{R}{F^2} \cdot \frac{(1 - F^{-1/\gamma})^{1/\kappa}}{(F^{1/\gamma} - 1)^2} \cdot \left\{ -1 + \kappa \cdot \left(F^{1/\gamma} + \gamma \cdot (F^{1/\gamma} - 1) \right) \right\}, \tag{16}$$

$$R = (1 - (1 - u_1)^\kappa)^{-\gamma - 1} (1 - u_1)^{\kappa - 1} (1 - (1 - u_2)^\kappa)^{-\gamma - 1} (1 - u_2)^{\kappa - 1}, \tag{17}$$

$$F = \left\{ -1 + (1 - (1 - u_1)^\kappa)^{-\gamma} + (1 - (1 - u_2)^\kappa)^{-\gamma} \right\}. \tag{18}$$

We apply the above two copulae as building blocks to construct two D-vine pair-copula systems for asymmetric dependency modelling.

The marginal distribution models I

- In the literature, AR threshold-GARCH models have been successfully employed by researchers to model stock market returns (see [Jondeau and Rockinger, 2006] and [Lai et al., 2009]). The AR-TGARCH models for the marginal distributions are defined by

$$r_t = \varphi_0 + \varphi_1 r_{t-1} + \varphi_2 r_{t-2} + \cdots + \varepsilon_t, \tag{19}$$

$$\varepsilon_t = \sigma_t z_t, \tag{20}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1^+ (\varepsilon_{t-1}^+)^2 + \alpha_1^- (\varepsilon_{t-1}^-)^2 + \beta_1 \sigma_{t-1}^2, \tag{21}$$

$$z_t \sim \mathrm{SkT}(\eta, \lambda). \tag{22}$$

The marginal distribution models II

- Eq. (22) specifies the residuals by using a Skewed Student t distribution, which is defined by [Hansen, 1994]:

$$\mathrm{SkT}(z;\eta,\lambda)=\begin{cases} bc(1+\frac{1}{\eta-2}(\frac{by+a}{1-\lambda})^2)^{-(\eta+1)/2}, & \text{if } y<-a/b \\ bc(1+\frac{1}{\eta-2}(\frac{by+a}{1+\lambda})^2)^{-(\eta+1)/2}, & \text{if } y\geq -a/b \end{cases} \tag{23}$$

where

$$a\equiv 4\lambda c\frac{\eta-2}{\eta-1},\quad b^2\equiv 1+3\lambda^2-a^2,\quad c\equiv\frac{\Gamma\frac{\eta+1}{2}}{\sqrt{\pi(\eta-2)}\Gamma(\frac{\eta}{2})}; \tag{24}$$

η denotes the DoF of the marginal distributions, λ represents the asymmetry parameter, and y follows the standard Student t distribution.

The asset allocation model I

The current asset allocation problem is a multi-period optimization problem. We adopt a monthly rolling window strategy to rebalance the portfolios.

- At each rebalance point, we update the parameters of the marginal distributions and the pair-copula systems.
- After updating the AR-TGARCH pari-copula system, the portfolio composition for next holding period is computed by optimizing a performance measure based on the simulated returns from the updated AR-TGARCH pari-copula system.
- Adjust the portfolio holdings based on the new portfolio composition and the previous one (transaction cost is not considered in this study).

The asset allocation model II

Three measures which are commonly used for the evaluation of risk-adjusted return of portfolios or strategies in finance are considered as the criterion candidates.

- The Omega ratio is proposed by [Keating and Shadwick, 2002]. One needs to split the portfolio returns $r_{p,t}$ into loss and gain above and below a given threshold r_d when computing the Omega ratio. The Omega is the ratio of the probability of having a gain by the probability of having a loss:

$$\Omega(r_{P,t}) = \frac{\int_{r_d}^{+\infty}(1 - G(x))dx}{\int_{-\infty}^{r_d} G(x)dx}, \tag{25}$$

  where G function represents a cumulative distribution function (CDF) of the portfolio return $r_{P,t}$. If the distribution is unknown, empirical distribution functions can be used for approximation.

The asset allocation model III

- The Sharpe ratio is a measure of the excess return per unit of risk. The measure is based on the first moment and the second moment of the return distribution (see [Sharpe, 1994]):

$$\mathrm{SR}(r_{P,t}) = \frac{E(r_{P,t}) - r_f}{\sigma_p}, \tag{26}$$

where $r_{P,t}$ represents the portfolio return in the evaluation period, r_f is a risk-free return or a desired return, and σ_p is the standard deviation of the portfolio returns.

The asset allocation model IV

- [Gregoriou and Gueyie, 2003] proposed the modified Sharpe, which is a ratio of the excess return divided by the modified VaR . The modified Sharpe considers the skewness and kurtosis of the return distribution based on the original Sharpe ratio:

$$\mathrm{MSR}(r_{P,t}) = \frac{E(r_{P,t}) - r_f}{MVaR}, \tag{27}$$

the modified VaR is then defined by

$$\mathrm{MVaR} = \mu_p - \left(z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)S_p + \frac{1}{24}(z_\alpha^3 - 3z_\alpha)K_p - \frac{1}{36}(2z_\alpha^3 - 5z_\alpha)S_p^2 \right) \sigma_p. \tag{28}$$

The asset allocation I

Let $\mathscr{F}$ denote the fitness function of the optimization problem, and the following notations are used to describe the problem:

T	the time of portfolio construction or rebalance
$n_{i,t}$	number of shares of the i-th equity invested at time t
P_t	market value of the portfolio at time t
$S_{i,t}$	per-share market value of the i-th equity at time t
$r_{P,t}$	portfolio return at time t
$\mathscr{C}_g$	the equity set of the portfolio
x_g^{ℓ}	minimum weight of each equity
x_g^{u}	maximum weight of each equity.

The asset allocation II

Thus, the model can be briefly described as follows:

$$\min_{\mathbf{n}} \mathscr{F} = \begin{cases} I. & -\mathrm{Omega}(r_{P,t}) \\ II. & -\mathrm{SR}(r_{P,t}) \\ III. & -\mathrm{MSR}(r_{P,t}) \end{cases}$$

subject to

$$r_{P,t} = \ln\left(\frac{P_t}{P_{t-1}}\right)$$

$$P_t = \sum_i n_{i,t} \cdot S_{i,t} \quad \text{for } i \in \mathscr{C}_g$$

$$n_{i,T} \in \mathbb{N}_0^+$$

$$x_g^\ell \leq \frac{n_{i,T} \cdot S_{i,T}}{P_T} \leq x_g^u \quad \text{for } i \in \mathscr{C}_g.$$

Differential Evolution is used to solved the above problem.

Data I

The portfolio comprises a set of equities $\mathscr{C}_g$ from the top 10 S&P 500 stocks: Johnson & Johnson (J&J), Cisco System, Bank of America (BoA), General Electric (GE) and AT&T, which are belonged to health care, information technology, finance, industrials and telecommunication sector respectively. The daily price the equities of the period January 2, 2000 to July 1, 2009 were downloaded from Datastream. The out-of-sample period starts from Jan. 2006 to July 2009.

Data II

Table: Summary statistics on the daily returns

	J&J	Cisco	BoA	GE	AT&T
Mean	0.0001	-0.0004	-0.0002	-0.0005	-0.0003
Max	0.1154	0.2182	0.3021	0.1798	0.1508
Min	-0.1725	-0.1405	-0.3421	-0.1368	-0.1354
SD	0.0146	0.0311	0.0361	0.0229	0.0208
Ske	-0.6018	0.3322	-0.2921	0.0915	0.1345
eKu	14.5283	5.1659	22.8774	7.2292	4.3939
Q(1)	22.5018	54.0363	237.3108	148.6092	31.0346
p-values	0.0000	0.0000	0.0000	0.0000	0.0000
Q(10)	262.1906	497.9050	1,377.3619	1,093.9157	559.5125
p-values	0.0000	0.0000	0.0000	0.0000	0.0000
LM(1)	22.4624	53.9455	236.8924	148.7618	30.9800
p-values	0.0000	0.0000	0.0000	0.0000	0.0000
LM(10)	145.3386	217.6292	472.3026	438.0548	257.3098
p-values	0.0000	0.0000	0.0000	0.0000	0.0000

The parameter estimates of the pair-copula systems

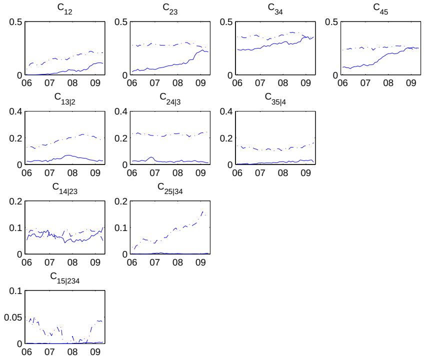


Figure: τ_L (solid lines) and τ_U (dash-dot lines) of the Joe-Clayton pair-copula system

The parameter estimates of the pair-copula systems

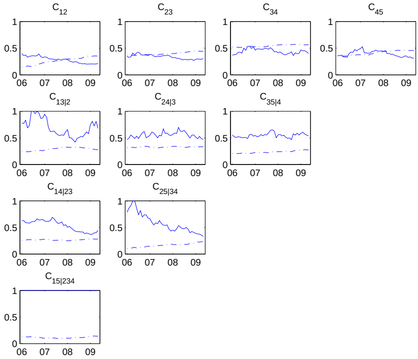


Figure: Standardized $\hat{\nu}$ (solid lines) and ρ (dash-dot lines) of the Student t pair-copula system

The parameter estimates of the pair-copula systems I

Table: Parameters of the Joe-Clayton (from the last rebalance period)

Joe-Clayton	τ_L	τ_U		τ_L	τ_U
C_{12}	0.1070	0.2080	$C_{24\|3}$	0.0130	0.2430
	(0.0014)	(0.0012)		(0.0001)	(0.0010)
C_{23}	0.2140	0.2530	$C_{35\|4}$	0.0170	0.1640
	(0.0015)	(0.0011)		(0.0005)	(0.0073)
C_{34}	0.3550	0.3480	$C_{14\|23}$	0.1010	0.0500
	(0.0011)	(0.0009)		(0.0011)	(0.0015)
C_{45}	0.2510	0.2430	$C_{25\|34}$	0.0030	0.1500
	(0.0013)	(0.0012)		(0.0000)	(0.0010)
$C_{13\|2}$	0.0280	0.1900	$C_{15\|234}$	0.0030	0.0390
	(0.0004)	(0.0008)		(0.0000)	(0.0009)
NLL	-877.8221				
χ^2 test:	statistic	8.9631	p-value	0.1757	

The parameter estimates of the pair-copula systems II

Table: Parameters of the Student t (from the last rebalance period)

Student t	ρ/τ	ν		ρ/τ	ν
C_{12}	0.3512/0.2284	6.3505	$C_{24\|3}$	0.3315/0.2151	14.1036
	(0.0006)	(1.3870)		(0.0006)	(24.4632)
C_{23}	0.4384/0.2889	9.1081	$C_{35\|4}$	0.2689/0.1733	16.1225
	(0.0005)	(5.2193)		(0.0006)	(>30)
C_{34}	0.5669/0.3837	12.0732	$C_{14\|23}$	0.2841/0.1834	13.3628
	(0.0003)	(15.2316)		(0.0007)	(18.3792)
C_{45}	0.4546/0.3005	9.4664	$C_{25\|34}$	0.2341/0.1504	9.9268
	(0.0005)	(6.4654)		(0.0007)	(5.7908)
$C_{13\|2}$	0.2797/0.1805	20.2949	$C_{15\|234}$	0.1427/0.0912	>30
	(0.0006)	(>30)		(0.0007)	(>30)
NLL	-976.7462				
χ^2 test:	statistic	11.9468	p-value	0.0632	

Analysis of portfolio performance I

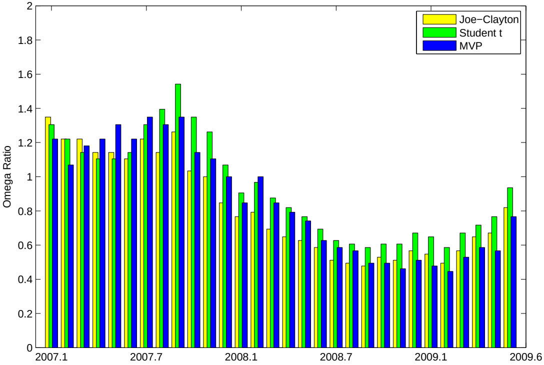


Figure: Out-of-sample Omega ratios

Analysis of portfolio performance II

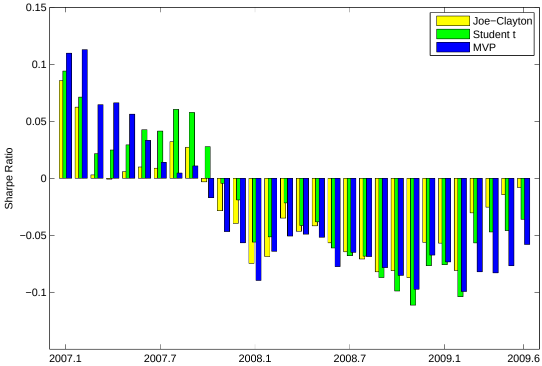


Figure: Out-of-sample Sharpe ratios

Analysis of portfolio performance III

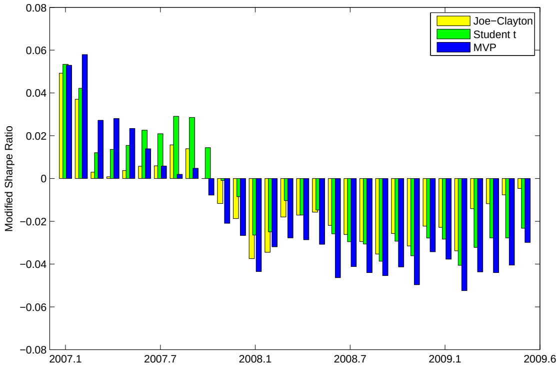
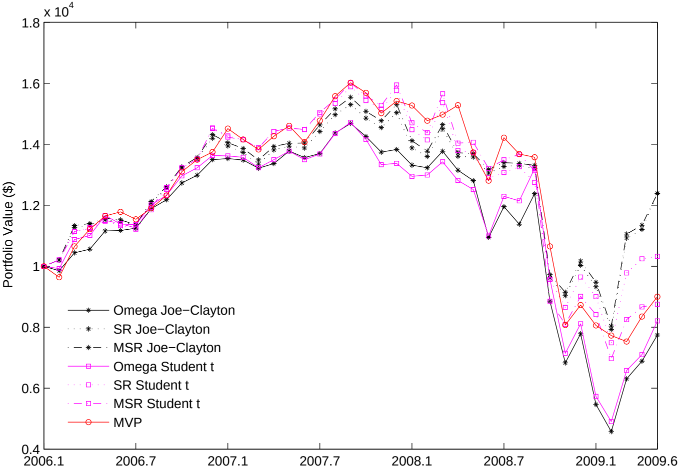


Figure: Out-of-sample modified Sharpe ratios

Analysis of portfolio performance IV



Conclusions I

- As the parameters in the Joe-Clayton pair-copula system indicate that, the lower rank correlation measure τ_L are smaller than the upper rank measure τ_U in the examined period. After converting the ρ in the Student t pair-copula system to Kendall's τ, it is found that the converted τ are always close to the upper rank correlation parameters τ_U of the corresponding pair-copula in the Joe-Clayton system.
- As the analysis of the out-of-sample portfolio performance shows that, the benefit from using the pair-copula models is not significant before the U.S. credit crunch while comparing to the benchmark portfolio. However, during the crisis and the market recovery, the Joe-Clayton copula system provides a better way to capture the positive return movements than the Student t copula system and the benchmark.

Conclusions II

- It is also found that, the portfolios constructed by maximizing the Omega ratio always yield the lowest wealth comparing to other portfolios, whereas the modified Sharpe ratio is suggested to be appropriate for the proposed asset allocation model.
- Future research may focus on the choice of individual pair-copula type in the vine structure for asset allocation problems. Moreover, whether a mixture vine pair-copula structure helps to improve the out-of-sample portfolio performance can be further studied.

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Thanks


I have no idea what you're talking about...
PYZAM.com
...so here's a bunny with a pancake on its head.